State-space models and the Kalman filter

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Let's start from an unobservable stochastic column vector with r elements $\boldsymbol{\xi}_1$, which is generated according to the following rule

$$\boldsymbol{\xi}_1 = \mathbf{F} \cdot \boldsymbol{\xi}_0 + \mathbf{v},$$

where $\boldsymbol{\xi}_0$ is a known vector of constants and $\mathbf{v} \sim R\left[0, \mathbf{Q}\right]$.
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Now enter a third observable vector with n elements, $\mathbf{y}$, of which we know the following:

$$\mathbf{y} = \mathbf{H}'\xi_1 + \mathbf{w}$$

where $\mathbf{w} \sim R\left[0, \mathbf{R}\right]$ ($\mathbf{R}$ is also known). For now, let's assume for simplicity that $\mathbf{v}$ and $\mathbf{w}$ are uncorrelated.

.

The GLS solution. Let's organise what we know in the following system of equations:

$$\begin{bmatrix} \widetilde{\xi}_1 \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{H}' \end{bmatrix} \xi_1 + \begin{bmatrix} -\mathbf{v} \\ \mathbf{w} \end{bmatrix}$$

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The covariance matrix for the "disturbance term" is known, so we can use GLS and compute

$$\begin{aligned} \widehat{\xi}_1 &= \left(\begin{bmatrix} \mathbf{I} & \mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{Q} & 0 \\ 0 & \mathbf{R} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I} \\ \mathbf{H}' \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{I} & \mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{Q} & 0 \\ 0 & \mathbf{R} \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{\xi}_1 \\ \mathbf{y} \end{bmatrix} = \\ &= \left[\mathbf{Q}^{-1} + \mathbf{H}\mathbf{R}^{-1}\mathbf{H}' \right]^{-1} \left[\mathbf{Q}^{-1} \widetilde{\xi}_1 + \mathbf{H}\mathbf{R}^{-1}\mathbf{y} \right] = \\ &= \left(\left[\mathbf{Q}^{-1} + \mathbf{H}\mathbf{R}^{-1}\mathbf{H}' \right]^{-1} \mathbf{Q}^{-1} \right) \widetilde{\xi}_1 + \left(\left[\mathbf{Q}^{-1} + \mathbf{H}\mathbf{R}^{-1}\mathbf{H}' \right]^{-1} \mathbf{H}\mathbf{R}^{-1} \right) \mathbf{y} \end{aligned} \tag{1}$$

(matrix-weighted average: nice)

Note: from GLS theory, we also know that

$$V(\widehat{\xi}_1) = \left(\mathbf{Q}^{-1} + \mathbf{H}\mathbf{R}^{-1}\mathbf{H}'\right)^{-1}.$$

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Now substitute into (1)

$$\left(\left[\mathbf{Q}^{-1} + \mathbf{H}\mathbf{R}^{-1}\mathbf{H}'\right]^{-1}\mathbf{Q}^{-1}\right)\widetilde{\xi_1} + \left(\left[\mathbf{Q}^{-1} + \mathbf{H}\mathbf{R}^{-1}\mathbf{H}'\right]^{-1}\mathbf{H}\mathbf{R}^{-1}\right)\left(\mathbf{H}'\widetilde{\xi_1} + \mathbf{e}\right)$$

and simplify to get

$$\widehat{\xi_1} = \widetilde{\xi_1} + \mathbf{K}\mathbf{e} \tag{2}$$

The "wise-up" algorithm

We could wrap up all of the above into an algorithm whose steps are:

1. given ξ_0, form a first guess of $\xi_1 = \widetilde{\xi}_1 = \mathbf{F}\xi_0$
2. use $\widetilde{\xi}_1$ to predict $\mathbf{y}$ via $\widehat{\mathbf{y}} = \mathbf{H}'\widetilde{\xi}_1$
3. observe $\mathbf{y}$
4. compute the forecast error on $\mathbf{y}$, $\mathbf{e} = \mathbf{y} - \widehat{\mathbf{y}}$
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Steps 1 and 2 don't require $\mathbf{y}$; if $\mathbf{y}$ is costly to acquire you can split the above as:

- forecast $\mathbf{y}$ (1–2)
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Let's call this the "wise-up" algorithm.

Generalisation

Now suppose we only observe ξ_0 imperfectly: all we have at the beginning is an unbiased estimate of ξ_0 (say, $\widehat{\xi_0}$) with a known covariance matrix (all would just be a limiting case with $V(\widehat{\xi_0}) = 0$).

Nothing would change, except for messier algebra.

This generalisation, however, paves the way to the possibility of iterating the algorithm above.
This is the intuition behind the celebrated *Kalman Filter*. But first, we need to define what a "state-space model" is.

A linear state-space model can be written as

$$\boldsymbol{\xi}_{t+1} = \mathbf{F}_t \boldsymbol{\xi}_t + \mathbf{v}_t \tag{3}$$

$$\mathbf{y}_t = \mathbf{A}_t' \mathbf{x}_t + \mathbf{H}_t' \boldsymbol{\xi}_t + \mathbf{w}_t \tag{4}$$

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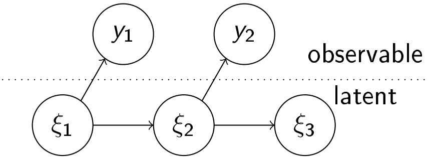

Figure: State space model through time

Usually, the $(r \times 1)$ vector $\mathbf{v}_t$ and the $(n \times 1)$ vector $\mathbf{w}_t$ are assumed to be vector white noise.

Let's start with something simple:

$$y_t = \varphi y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1};$$

now invert $A(L)$ to give $y_t = C(L)a_t$, where $a_t = A(L)^{-1}\varepsilon_t$, or

$$a_t = \varphi a_{t-1} + \varepsilon_t \tag{5}$$

Therefore,

$$y_t = a_t + \theta a_{t-1}. \tag{6}$$

It's not difficult to see that

- the only thing we observe is the scalar y_t; therefore, $\mathbf{y}_t = y_t$.
- To express y_t, you need *both* a_t and its lag; therefore, $\boldsymbol{\xi}_t$ must contain at least a_t and a_{t-1}. Using (6),

$$y_t = [1 \quad \theta] \begin{bmatrix} a_t \\ a_{t-1} \end{bmatrix}$$

- $\boldsymbol{\xi}_{t+1}$ can be written as a function of itself lagged, and a white-noise process:

$$\begin{bmatrix} a_{t+1} \\ a_t \end{bmatrix} = \begin{bmatrix} \varphi & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_t \\ a_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}$$

So the ARMA(1,1) model can be written in state-space form this way:

$$\mathbf{H} = \begin{bmatrix} 1 \\ \theta \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} \varphi & 0 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{Q} = \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Not difficult to generalise to ARMAs of any order.
State-space representations are, in general, not unique. For example, there are other ways to express an ARMA(1,1):

$$\mathbf{H} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} \varphi & \theta \\ 0 & 0 \end{bmatrix} \qquad \mathbf{Q} = \sigma^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

or

$$\mathbf{H} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} \varphi & 0 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{Q} = \sigma^2 \begin{bmatrix} 1 & \theta \\ \theta & \theta^2 \end{bmatrix}.$$

The best one to use depends on several factors, not least computational efficiency.

The Kalman recursions

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The forward pass is mostly used in estimating the parameters of the models.

Smoothing is used, instead, to reconstruct the historical time path of the state vector.

We start from a preliminary estimate of the state at time 0: a random variable with known mean and variance:

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We may (a) guess the value of $\mathbf{y}_0$ via equation (4) and of ξ_1 via equation (3) (b) when $\mathbf{y}_1$ becomes available, update our guess on ξ_1 wia the "wise-up" algorithm.

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So we have

1. a forecast error $\mathbf{e}_1$ with its own covariance matrix $\mathbf{\Sigma}_1$;
2. a new starting point $\tilde{\xi}_1$ with its own covariance matrix $\mathbf{P}_1$;

which means that we can repeat the algorithm from ξ_1, so to obtain $\mathbf{e}_2$ and $\mathbf{\Sigma}_2$, and then on and on, until our data end, and we stop at ξ_T, $\mathbf{e}_T$ and $\mathbf{\Sigma}_T$.

The formulae for performing the above are generalisations of (2) and (3) that can be found in specialised books.

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They're messier because:

1. The state matrices can be time-varying;
2. the disturbances $\mathbf{w}_t$ and $\mathbf{v}_t$ may not be uncorrelated, and
3. initialisation for the algorithm may not be obvious: notably, sometimes you may want to endow ξ_0 with an "infinite" covariance matrix (this case is known as the "diffuse prior" case).

One last thing on the filtering pass: if the parameters of the system are the true ones, filtering gives you the best guess of the state vector $\boldsymbol{\xi}_t$, given the information up to time $t-1$:

$$\widehat{\boldsymbol{\xi}}_t = E(\boldsymbol{\xi}_t | \Im_{t-1})$$

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Therefore, the one-step-ahead prediction errors give you

$$\mathbf{e}_t = \mathbf{y}_t - E(\mathbf{y}_t | \Im_{t-1})$$

and by construction $E(\mathbf{e}_t | \Im_{t-1}) = 0$ (martingale difference sequence).

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Therefore, you can calculate the Gaussian log-density for each of them:

$$\ell_t = \text{const} \; - \frac{1}{2} \ln |\boldsymbol{\Sigma}_t| - \frac{1}{2} \mathbf{e}_t' \boldsymbol{\Sigma}_t^{-1} \mathbf{e}_t \tag{7}$$

and $\sum_t \ell_t$ may be interpreted as a log-likelihood. This is the main idea behind the usage of the Kalman Filter as an inferential tool.

Once you have the log-likelihood for a given set of system matrices, you can either

- maximize it (as a rule, numerically) and go for classical (frequentist) inference methios
- go Bayesian, use it to update your priors and get your posteriors, which is what DSGE people do.

Smoothing

Basic idea:
There are several smoothing techniques. What people usually do in econom(etr)ics is *fixed-interval* smoothing.

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Basically, you use the "wise-up" algorithm backwards (algebra is complicated, though) and start from $\hat{\boldsymbol{\xi}}_T$ and use $\mathbf{e}_{T-1}, \mathbf{e}_{T-2}, \ldots$ to work your way backwards to $\boldsymbol{\xi}_0$.

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Again, see the books for details.

Random walk plus noise

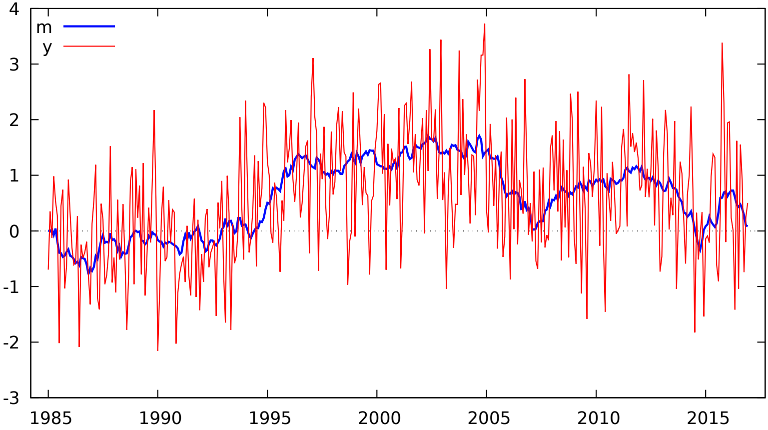
Let us generate a "random walk plus noise" artificial process

$$\begin{aligned} m_t &= m_{t-1} + u_t \\ y_t &= m_t + \varepsilon_t \end{aligned}$$

where $V(u_t) = 0.01$ and $V(\varepsilon_t) = 1$; gretl code follows

```
nulldata 384
set seed 71218
setobs 12 1985:1

series m = cum(normal()*0.1)
series y = m + normal()
```

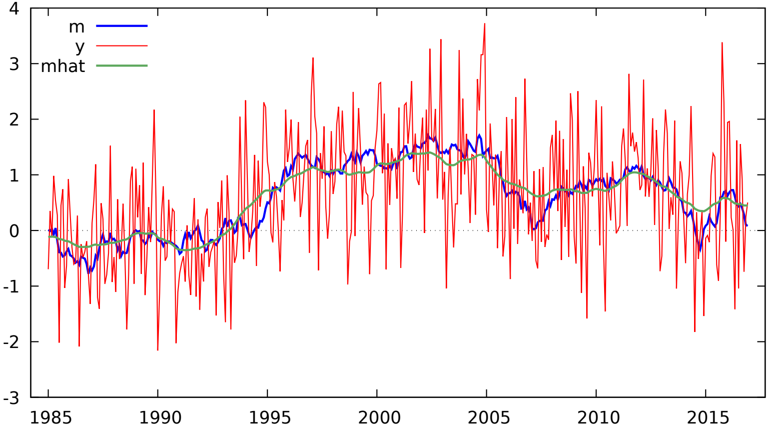


Note that in practice, the only observable series would be y_t (the red line). What we want to do is use the Kalman filter to recover m_t.

```
matrix F = 1
matrix H = 1
matrix Q = 0.01
mod = ksetup(y, F, H, Q)
mod.diffuse = 1
mod.obsvar = 1

err = ksmooth(&mod)
series mhat = mod.state
```

produces


m
y
mhat
4
3
2
1
0
-1
-2
-3
1985
1990
1995
2000
2005
2010
2015

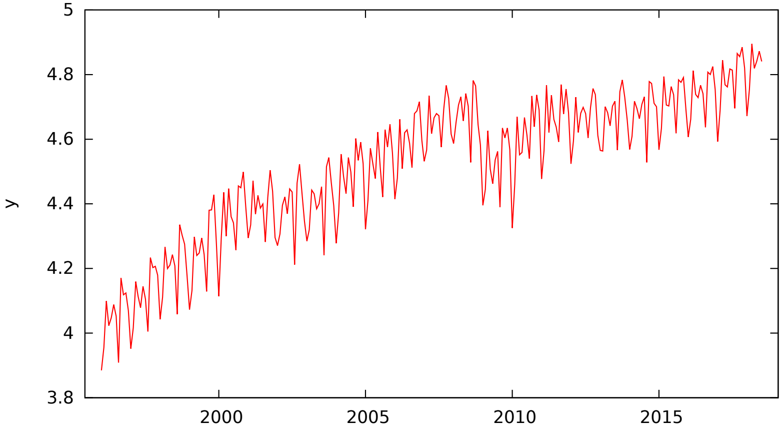
Structural Time Series Models

Invented by the English statistician Andrew Harvey in the 1980s.
His approach resembles what guitarists do with pedal effects: in order to get the features they want, they stack effects on top of one another, until the sound is right.

Model name	Formula
Random walk plus noise	$\mu_t = \mu_{t-1} + \varepsilon_t$ $y_t = \mu_t + u_t$
Local linear trend	$\mu_t = \mu_{t-1} + \beta_{t-1} + \varepsilon_t$ $\beta_t = \beta_{t-1} + \eta_t$ $y_t = \mu_t + u_t$
Trigonometric Cycle	$a_t = \rho\left(\cos\theta \cdot a_{t-1} + \sin\theta \cdot b_{t-1}\right) + \omega_{1,t}$ $b_t = \rho\left(-\sin\theta \cdot a_{t-1} + \cos\theta \cdot b_{t-1}\right) + \omega_{2,t}$ $y_t = a_t$
$\vdots$	$\vdots$

Example: Austrian IP

```
open dbnomics
data IMF/IFS/M.AT.AIPMA_IX --name=Austria_IP
series y = log(Austria_IP)
include StrucTiSM.gfn
mod = STSM_GUImeta(y, 3, null, 0, 1, 1, 0, 1)
```


5
4.8
4.6
4.4
4.2
4
3.8
y
2000
2005
2010
2015

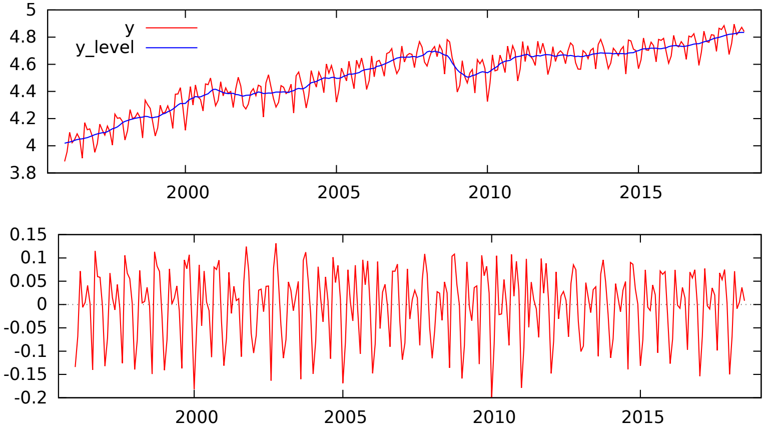
```
Structural model for y, 1996:01 - 2018:07 (T = 271)

                     coefficient   std. error      z       p-value
  ---------------------------------------------------------------
  Irregular          0.0123792     0.00895181    1.383    0.1667
  Trend              0.0131580     0.00144715    9.092    9.69e-20 ***
  Slope              0.000181810   0.000215125   0.8451   0.3980
  Seasonal (dums)    0.0196931     0.00417469    4.717    2.39e-06 ***

Average log-likelihood = 1.51121

Specification:

Stochastic trend, stochastic slope, dummy seasonals (stoch.),
irregular component
```


5
4.8
4.6
4.4
4.2
4
3.8
y
y_level
2000
2005
2010
2015
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
2000
2005
2010
2015