State-space models and the Kalman filter

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Let's start from an unobservable stochastic column vector with \( r \) elements \( \xi_1 \), which is generated according to the following rule

\[
\xi_1 = F \cdot \xi_0 + v,
\]

where \( \xi_0 \) is a known vector of constants and \( v \sim R[0, Q] \). Our best guess on \( \xi_1 \) is \( \tilde{\xi}_1 \equiv F \xi_0 \).
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Now enter a third observable vector with \( n \) elements, \( y \), of which we know the following:

\[
y = H'\xi_1 + w
\]

where \( w \sim R[0, R] \) (\( R \) is also known). For now, let’s assume for simplicity that \( v \) and \( w \) are uncorrelated.
The GLS solution. Let’s organise what we know in the following system of equations:

\[
\begin{bmatrix}
\tilde{\xi}_1 \\
y
\end{bmatrix} =
\begin{bmatrix}
I \\
H'
\end{bmatrix} \xi_1 +
\begin{bmatrix}
-v \\
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w \end{bmatrix}
\]

The covariance matrix for the “disturbance term” is known, so we can use GLS and compute

\[
\hat{\xi}_1 = \left( \begin{bmatrix} I & H \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}^{-1} \begin{bmatrix} I \\ H' \end{bmatrix} \right)^{-1} \begin{bmatrix} I & H \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\xi}_1 \\
y \end{bmatrix} =
\]

\[
= \left[ Q^{-1} + HR^{-1}H' \right]^{-1} \left[ Q^{-1} \tilde{\xi}_1 + HR^{-1}y \right] =
\]

\[
= \left( \left[ Q^{-1} + HR^{-1}H' \right]^{-1} Q^{-1} \right) \tilde{\xi}_1 + \left( \left[ Q^{-1} + HR^{-1}H' \right]^{-1} HR^{-1} \right) y
\]

(1)

(matrix-weighted average: nice)
Note: from GLS theory, we also know that

\[ V(\hat{\xi}_1) = \left( Q^{-1} + HR^{-1}H' \right)^{-1}. \]
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$$V(\hat{\xi}_1) = (Q^{-1} + HR^{-1}H')^{-1}.$$  

Even nicer: try to forecast $y$ on the basis of $\xi_0$. Best choice is $\hat{y} = H'\tilde{\xi}_1$. The forecast error $e$, satisfies $y = H'\tilde{\xi}_1 + e$. (Covariance matrix for $e$ is also easy to compute; let’s just call it $\Sigma$. Useful later)
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Now substitute into (1)

$$\left( [Q^{-1} + HR^{-1}H']^{-1} Q^{-1} \right) \tilde{\xi}_1 + \left( [Q^{-1} + HR^{-1}H']^{-1} HR^{-1} \right) \left( H'\tilde{\xi}_1 + e \right)$$

and simplify to get

$$\hat{\xi}_1 = \tilde{\xi}_1 + Ke \quad (2)$$
The “wise-up” algorithm

We could wrap up all of the above into an algorithm whose steps are:

1. given $\xi_0$, form a first guess of $\xi_1 = \hat{\xi}_1 = F\xi_0$
2. use $\hat{\xi}_1$ to predict $y$ via $\hat{y} = H'\hat{\xi}_1$
3. observe $y$
4. compute the forecast error on $y$, $e = y - \hat{y}$
5. use $e$ to update our guess on $\xi_1$ via $\hat{\xi}_1 + Ke$. 

$K$ is called the gain matrix.
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Steps 1 and 2 don’t require $y$; if $y$ is costly to acquire you can split the above as:

- forecast $y$ (1–2)
- once $y$ is observed, update your estimate of $\xi_1$ (3–5) via $e$. 

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Let’s call this the “wise-up” algorithm.
Generalisation

Now suppose we only observe $\xi_0$ imperfectly: all we have at the beginning is an unbiased estimate of $\xi_0$ (say, $\hat{\xi}_0$) with a known covariance matrix (all would just be a limiting case with $V(\hat{\xi}_0) = 0$).

Nothing would change, except for messier algebra.

This generalisation, however, paves the way to the possibility of iterating the algorithm above.

This is the intuition behind the celebrated *Kalman Filter*. But first, we need to define what a “state-space model” is.
A linear state-space model can be written as

$$\xi_{t+1} = F_t \xi_t + v_t \quad (3)$$

$$y_t = A'_t x_t + H'_t \xi_t + w_t \quad (4)$$
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\[
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\]

(3) \hspace{1cm} (4)

Usually, the \((r \times 1)\) vector \(v_t\) and the \((n \times 1)\) vector \(w_t\) are assumed to be vector white noise.

**Figure:** State space model through time
Let’s start with something simple:

\[ y_t = \varphi y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}; \]

now invert \( A(L) \) to give \( y_t = C(L)a_t \), where \( a_t = A(L)^{-1}\varepsilon_t \), or

\[ a_t = \varphi a_{t-1} + \varepsilon_t \quad (5) \]

Therefore,

\[ y_t = a_t + \theta a_{t-1}. \quad (6) \]
It’s not difficult to see that

- the only thing we observe is the scalar $y_t$; therefore, $y_t = y_t$.
- To express $y_t$, you need both $a_t$ and its lag; therefore, $\xi_t$ must contain at least $a_t$ and $a_{t-1}$. Using (6),

$$y_t = \begin{bmatrix} 1 & \theta \end{bmatrix} \begin{bmatrix} a_t \\ a_{t-1} \end{bmatrix}$$

- $\xi_{t+1}$ can be written as a function of itself lagged, and a white-noise process:

$$\begin{bmatrix} a_{t+1} \\ a_t \end{bmatrix} = \begin{bmatrix} \varphi & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_t \\ a_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}$$

So the ARMA(1,1) model can be written in state-space form this way:

$$H = \begin{bmatrix} 1 \\ \theta \end{bmatrix} \quad F = \begin{bmatrix} \varphi & 0 \\ 1 & 0 \end{bmatrix} \quad Q = \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
Not difficult to generalise to ARMA{s} of any order.
State-space representations are, in general, not unique. For example, there are other ways to express an ARMA(1,1):

\[
H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad F = \begin{bmatrix} \varphi & \theta \\ 0 & 0 \end{bmatrix}, \quad Q = \sigma^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},
\]

or

\[
H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad F = \begin{bmatrix} \varphi & 0 \\ 1 & 0 \end{bmatrix}, \quad Q = \sigma^2 \begin{bmatrix} 1 & \theta \\ \theta & \theta^2 \end{bmatrix}.
\]

The best one to use depends on several factors, not least computational efficiency.
The Kalman recursions

Essentially, a clever way to apply the “wise-up” algorithm to state-space models.
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In practice, we perform two “passes”:

1. the “forward” pass (which is the Kalman filter proper),
2. the “backwards” pass, also known as “smoothing”.
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1. the “forward” pass (which is the Kalman filter proper),
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The forward pass is mostly used in estimating the parameters of the models.

Smoothing is used, instead, to reconstruct the historical time path of the state vector.
We start from a preliminary estimate of the state at time 0: a random variable with known mean and variance:

\[ \xi_0 \sim R(\tilde{\xi}_0, \mathbf{P}_0) \]
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We may (a) guess the value of $y_0$ via equation (4) and of $\xi_1$ via equation (3) (b) when $y_1$ becomes available, update our guess on $\xi_1$ via the “wise-up” algorithm.
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So we have

1. a forecast error $e_1$ with its own covariance matrix $\Sigma_1$;
2. a new starting point $\tilde{\xi}_1$ with its own covariance matrix $P_1$;

which means that we can repeat the algorithm from $\xi_1$, so to obtain $e_2$ and $\Sigma_2$, and then on and on, until our data end, and we stop at $\xi_T$, $e_T$ and $\Sigma_T$. 
The formulae for performing the above are generalisations of (2) and (3) that can be found in specialised books.
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1. The state matrices can be time-varying;
2. the disturbances $\mathbf{w}_t$ and $\mathbf{v}_t$ may not be uncorrelated, and
3. initialisation for the algorithm may not be obvious: notably, sometimes you may want to endow $\xi_0$ with an “infinite” covariance matrix (this case is known as the “diffuse prior” case).
One last thing on the filtering pass: if the parameters of the system are the true ones, filtering gives you the best guess of the state vector $\xi_t$, given the information up to time $t - 1$:

$$\hat{\xi}_t = E(\xi_t | \mathcal{F}_{t-1})$$
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Therefore, the one-step-ahead prediction errors give you

$$e_t = y_t - E(y_t | \mathcal{F}_{t-1})$$

and by construction $E(e_t | \mathcal{F}_{t-1}) = 0$ (martingale difference sequence).
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Therefore, you can calculate the Gaussian log-density for each of them:

$$\ell_t = \text{const} - \frac{1}{2} \ln |\Sigma_t| - \frac{1}{2} e_t' \Sigma_t^{-1} e_t$$

and $\sum_t \ell_t$ may be interpreted as a log-likelihood. This is the main idea behind the usage of the Kalman Filter as an inferential tool.
Once you have the log-likelihood for a given set of system matrices, you can either

- maximize it (as a rule, numerically) and go for classical (frequentist) inference methods
- go Bayesian, use it to update your priors and get your posteriors, which is what DSGE people do.
Smoothing

Basic idea:
There are several smoothing techniques. What people usually do in econom(etr)ics is *fixed-interval* smoothing.
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Basically, you use the “wise-up” algorithm backwards (algebra is complicated, though) and start from $\hat{x}_T$ and use $e_{T-1}, e_{T-2}, \ldots$ to work your way backwards to $x_0$. 
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Basically, you use the “wise-up” algorithm backwards (algebra is complicated, though) and start from $\hat{\xi}_T$ and use $e_{T-1}, e_{T-2}, \ldots$ to work your way backwards to $\xi_0$.
Again, see the books for details.
Random walk plus noise

Let us generate a “random walk plus noise” artificial process

\[ m_t = m_{t-1} + u_t \]
\[ y_t = m_t + \varepsilon_t \]

where \( V(u_t) = 0.01 \) and \( V(\varepsilon_t) = 1 \); gretl code follows

nulldata 384
set seed 71218
setobs 12 1985:1

series m = cum(normal()*0.1)
series y = m + normal()
Note that in practice, the only observable series would be $y_t$ (the red line). What we want to do is use the Kalman filter to recover $m_t$. 
matrix F = 1
matrix H = 1
matrix Q = 0.01
mod = ksetup(y, F, H, Q)
mod.diffuse = 1
mod.obsvar = 1

err = ksmooth(&mod)
series mhat = mod.state

produces
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An example

-3 -2 -1 0 1 2 3
m y mhat

-4
-3
-2
-1
0
1
2
3
4

m
y
mhat
Invented by the English statistician Andrew Harvey in the 1980s. His approach resembles what guitarists do with pedal effects: in order to get the features they want, they stack effects on top of one another, until the sound is right.
<table>
<thead>
<tr>
<th>Model name</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Random walk plus noise</td>
<td>$\mu_t = \mu_{t-1} + \varepsilon_t$</td>
</tr>
<tr>
<td>Local linear trend</td>
<td>$\mu_t = \mu_{t-1} + \beta_{t-1} + \varepsilon_t$</td>
</tr>
<tr>
<td>Trigonometric Cycle</td>
<td>$a_t = \rho (\cos \theta \cdot a_{t-1} + \sin \theta \cdot b_{t-1}) + \omega_{1,t}$</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example: Austrian IP

open dbnomics
data IMF/IFS/M.AT.AIPMA_IX --name=Austria_IP
series y = log(Austria_IP)
include StrucTiSM.gfn
mod = STSM_GUImeta(y, 3, null, 0, 1, 1, 0, 1)
### Structural model for y, 1996:01 - 2018:07 (T = 271)

<table>
<thead>
<tr>
<th></th>
<th>Coefficient</th>
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<th>z</th>
<th>p-value</th>
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<tr>
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<tr>
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<td>0.00144715</td>
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<td>9.69e-20 ***</td>
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<tr>
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<td>0.000215125</td>
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<td>0.3980</td>
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<tr>
<td>Seasonal (dums)</td>
<td>0.0196931</td>
<td>0.00417469</td>
<td>4.717</td>
<td>2.39e-06 ***</td>
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</tbody>
</table>

Average log-likelihood = 1.51121

**Specification:**

Stochastic trend, stochastic slope, dummy seasonals (stoch.), irregular component
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